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THE MULTIVARIATE NORMAL BEHAVIOUR OF A SYMMETRIC m-DIMENSIONAL SIMPLE EPIDEMIC

by

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ABSTRACT

We consider a group of n susceptible individuals who are exposed to m contagious diseases. The progress of the epidemic among the individuals is modeled by a stochastic process $\underset{\sim}{X}_n(t) = (X_{n,1}(t), \ldots, X_{n,m}(t))$, t in $(0, \infty)$. The components of $\underset{\sim}{X}_n(t)$ describe the number of infective individuals with the respective disease at time t.

For a class of epidemic models named symmetric m-dimensional simple epidemics [Billard, Lacayo and Langberg (1979)] we establish, with a suitable standartization, the asymptotic normal convergence of $\underset{\sim}{X}_n(t)$ as $n \to \infty$ for t in $(0, \infty)$.

1. Introduction and Summary.

In a simple epidemic situation we assume that a population of susceptible individuals (*susceptibles*) is exposed only to one contagious disease (*disease*) [Bailey (1975)]. However, frequently susceptibles are exposed simultaneously to more than one disease, as is the case with different types of flu. In this paper we consider a population of susceptibles exposed to m diseases. We say that the population of susceptibles undergoes an *m-dimensional simple epidemic* if the following five assumptions hold.

(1.1) Each susceptible contracts at most one disease.

(1.2) Once a susceptible contracts disease r, r = 1, ..., m he remains contagious during the duration of the epidemic.

(1.3) An infective individual (*infective*) with disease r, r = 1, ..., m can transmit only that disease.

(1.4) Individuals neither join nor do they depart from the population, and

(1.5) At each point in time at most one susceptible contracts a disease.

Let T_0 denote the first time we have at least one infective with each of the various diseases, and let n denote the number of susceptibles at T_0. We describe the progress of an m-dimensional simple epidemic among susceptibles by an m-dimensional stochastic process $\underline{X}_n(t) = (X_{n,1}(t), \ldots, X_{n,m}(t))$. The components of $\underline{X}_n(t)$ represent the number of infectives with the respective diseases at time t measured from T_0.

Billard, Lacayo and Langberg (1979) (*BLL*) considered a special case of an m-dimensional simple epidemic and named it the *symmetric m-dimensional simple epidemic*. We say that a population of susceptibles undergoes a symmetric m-dimensional simple epidemic if the transition rates of disease 1 through m at time t, $t \in [0,\infty)$, rigorously defined in Section 2, are respectively given by:

(1.6) $n^{-1} a X_{n,r}(t)\{n - \sum_{r=1}^{m}(X_{n,r}(t) - X_{n,r}(0))\}$, $r = 1, \ldots, m$, $a \in (0,\infty)$.

The transition rates given by Equation (1.6) reflect (a) that all interactions between a susceptible and an infective are "equally likely" and (b) that not "too many" infectives are added in short time intervals; so that the duration time of the epidemic does not tend to zero as $n \to \infty$. We note that the symmetric m-dimensional simple epidemic generalizes the simple epidemic model used by McNeil (1972). In Section 2 we construct an m-dimensional stochastic process that describes the progress of a symmetric m-dimensional simple epidemic.

Let $\underline{X}(t) = \{X_1(t), \ldots, X_m(t)\}$, $t \in [0,\infty)$ be a random vector (rve) with independent components. Further, let the random variable (rva) $X_r(t)$ have a negative binomial distribution with parameters e^{-at} and $b_r \in \{1, 2, \ldots\}$ for $r = 1, \ldots, m$ and for $t \in [0,\infty)$. BLL (1979) assume that n is large compared to each $X_{n,r}(0)$, $r = 1, \ldots, m$, a condition formally given by,

(1.7) $$\lim_{n\to\infty} X_{n,r}(0) = b_r, \quad r = 1, \ldots, m.$$

They prove that $P\{X_{n,r}(t) - X_{n,r}(0) = k_r,\ r = 1, \ldots, m\}$ can be approximated by $\prod_{r=1}^{m} \binom{b_r + k_r - 1}{k_r} e^{-ab_r t}\left(1 - e^{-at}\right)^{k_r}$ for all $t \in [0,\infty)$, and state it formally as follows:

(1.8) The rve $\underline{X}_n(t)$ converges in distribution as $n \to \infty$ (converges) to the rve $\underline{X}(t)$ for all $t \in [0,\infty)$.

Further, BLL (1979) prove that

(1.9) the β^{th} moment of $X_{n,r}(t)$ converges as $n \to \infty$ to the β^{th} moment of $X_r(t)$ for all t, $\beta \in (0,\infty)$ and for $r = 1, \ldots, m$.

Thus, in particular one can approximate $EX_{n,r}(t)$ and $Var\{X_{n,r}(t)\}$ by $b_r(e^{\alpha t}-1)$ and by $b_r(e^{2\alpha t}-e^{\alpha t})$ respectively for $r = 1, \ldots, m$ and for $t \in (0,\infty)$.

It is quite conceivable that the epidemic starts with a burst of infectives with the different diseases. If this is the case Condition (1.7) does not necessarily hold; consequently the approximations discussed in the previous paragraph are not appropriate. To accomodate this situation we assume, in constrast to Condition (1.7), that the number of infectives at T_0 with the various diseases are proportional to n. For technical reasons we require a bit more and assume that for $r = 1, \ldots, m$

(1.10) $$\lim_{n\to\infty} (n^{-1}X_{n,r}(0)-\lambda_r)\sqrt{n} = 0, \text{ where } \lambda_1, \ldots, \lambda_m \in (0,\infty).$$

We note that the univariate version of Condition (1.10) was assumed, at least implicitly, by McNeil (1972) to obtain his asymptotic result. In Section 5 we show that under Condition (1.10) the rve $\underline{X}_n(t)$, with the suitable standardization, converges to a multivariate normal (MVN) rve for $t \in (0,\infty)$. One can use this result to approximate the state probability: $P\{X_{n,r}(t)-X_{n,r}(0)=k_r,\ r=1,\ldots,m\}$ for all $t \in (0,\infty)$ and all $k_1, \ldots, k_m \in \{0,1,\ldots\}$.

Let $X_n(t) = \sum_{r=1}^{m} X_{n,r}(t)$, $t \in [0,\infty)$, and let $L(n)$, $n = 1, 2, \ldots$, be a sequences of integers in the set $\{1,\ldots,n\}$ for almost all n, $n = 1, 2, \ldots$, respectively. Further, let $\xi_{n,k}$, $k = 1,\ldots,n$ be rva's assuming values in the set $\{1,\dots,m\}$ designating the disease responsible for the $X_n(0) + k$ infection respectively, and let I be the indicator function. In Section 5 we show that the rve $\{L^{-1}(n)\sum_{q=1}^{L(n)} I(\xi_{n,q}=r), r=1,\dots,m\}$ converges, with the suitable standardization, to a MVN rve provided

$$\lim_{n\to\infty} n^{-1}L(n) = z\in(0,1]. \tag{1.11}$$

One can use this result to approximate the probability of having simultaneously $X_{n,r}(0) + k_r$ infectives with disease r, $r = 1, \dots, m$ when the total number of infectives: $X_n(0) + L(n)$ is "almost" equal to a proportion of n.

In Sections 3 and 4 we present some lemmas needed in the proofs of the two main results of Section 5.

2. Model Construction.

In this section we construct an m-dimensional stochastic process that describes the progress of the symmetric m-dimensional simple epidemic among the susceptibles. We need some notation.

Let $T_{n,k}$, $k = 1, \ldots, n$ be the $k^{\underline{th}}$ interinfection-time defined as the time that elapses between the $X_n(0) + k - 1$ and the $X_n(0) + k$ infection. Let $S_{n,0} = 0$, let $S_{n,k} = \sum_{q=1}^{k} T_{n,q}$, $k = 1, \ldots, n$, and let $S_{n,n+1} = \infty$. Further, let I be the indicator function, and let $J_{k,r}$, $k = 1, \ldots, n$, $r = 1, \ldots, m$ be the index set of all infectives with disease r at the time $T_o + S_{n,k-1}$. Finally, let $\eta_{i,j,k}$, $i = 1, \ldots, X_n(0) + k - 1$, $j = 1, \ldots, n - k + 1$, $k = 1, \ldots, n$ be i.i.d exponential rva's defined on some probability space with a mean equal to $n\alpha^{-1}$.

Throughout we assume that the rva's $\eta_{i,j,k}$ describe the time measured from the $X_n(0) + k - 1$ infection, $k = 1, \ldots, n$ until the $i^{\underline{th}}$ contagious individual, $i = 1, \ldots, X_n(0) + k - 1$ causes the $j^{\underline{th}}$ susceptible, $j = 1, \ldots, n - k + 1$ to become the $k^{\underline{th}}$ infective.

We are ready now to construct the desired stochastic process. Let $k = 0, \ldots, n$, let $r = 1, \ldots, m$, and let $t \in [0,\infty)$. Then the following event equality holds.

$$(2.1) \qquad (X_{n,r}(t) - X_{n,r}(0) \geq k) = \bigcup_{q=k}^{n} (S_{n,q} \leq t < S_{n,q+1}, \sum_{j=1}^{q} I(\xi_{n,j} = r) \geq k).$$

Thus, to construct the process $\underline{X}_n(t)$ it suffices by Statement (2.1) to determine the distribution function of the rve $(T_{n,1}, \xi_{n,1}, \ldots, T_{n,n}, \xi_{n,n})$. We determine the distribution function of this rve in the following two lemmas.

Lemma 2.1. Let $r = 1, \ldots, m$. Then

(2.2) $$P\{\xi_{n,1}=r\} = X_{n,r}(0)\ X_n^{-1}\ (0),$$ and

(2.3) $$P\{\xi_{n,k}=r|\xi_{n,1},\ldots,\xi_{n,k-1}\} =$$
$$= \{X_{n,r}(0)+\sum_{q=1}^{k-1} I(\xi_{n,q}=r)\}\{X_n(0)+k-1\}^{-1},\ k = 2,\ \ldots,\ n.$$

Proof. Let $k = 1, \ldots, n$. Then

(2.4) $$(\xi_{n,k}=r) = (\min\{\eta_{i,j,k}:i\epsilon J_{k,r},j=1,\ldots,n-k+1\}<$$
$$< \min\{\eta_{i,j,k}:i\epsilon \bigcup_{\substack{e=1\\e\neq r}}^{m} J_{k,e},j=1,\ldots,n-k+1\}).$$

Consequently Statements (2.2) and (2.3) follow by simple calculations. ||

Lemma 2.2. The conditional rve $\{T_{n,1},\ldots,T_{n,n}\}|\xi_{n,1},\ldots,\xi_{n,n}$ has exponential independent components with means respectively equal to $n\ \alpha^{-1}\{(n-q+1)(X_n(0)+q-1)\}^{-1}$, $q = 1, \ldots, n$.

Proof. Let $k = 1, \ldots, n$. Then

(2.5) $$T_{n,k} = \min\{\eta_{i,j,k}:i=1,\ldots,X_n(0)+k-1,j=1,\ldots,n-k+1\}.$$

Consequently the result of the lemma follows. ||

Now, we show that the stochastic process constructed in the previous paragraph describes the progress of a symmetric m-dimensional simple epidemic. For the sake of completeness we first, present a definition of the transition rates of the various diseases.

Definition 2.3. Let $r = 1, \ldots, m$ and let $t\epsilon[0,\infty)$. Then the transition rate of disease r at time t is given by:

$$\lim_{h\to o^+} h^{-1}\ P\{X_{n,r}(t+h)-X_{n,r}(t)=1|\underline{X}_n(t)\}.$$

By the memoryless property of exponential rva's [Barlow, Proschan (1975), p.56] and by Statements (2.1), (2.4) and (2.5) it follows that the transition rates of the various diseases satisfy Equation (1.6).

Finally, for reference purposes we present the following lemma.

<u>Lemma 2.4.</u> (a) The rve's $\{T_{n,1}, \ldots, T_{n,n}\}$ and $\{\xi_{n,1}, \ldots, \xi_{n,n}\}$ are independent, and (b) The rva's $T_{n,1}, \ldots, T_{n,n}$ are independent.

<u>Proof.</u> The results of the lemma follow clearly from Lemma 2.2. ||

3. Preliminaries.

In this section we establish the asymptotic behaviour of a specific sequence of random vectors with Dirichlet distributions. We later use this result in Section 5.

For the sake of completeness we define the Dirichlet and the Gamma distributions.

Definition 3.1. Let $a_1, \ldots, a_\ell$ be positive real numbers, let $\Gamma(\theta) = \int_0^\infty e^{-x}x^{\theta-1}dx, \theta \in (0,\infty)$, and let $A = \{(x_1,\ldots,x_{\ell-1}): x_r > 0, r=1,\ldots,\ell-1, \sum_{r=1}^{\ell-1} x_r < 1\}$. We say that the rve $(W_1,\ldots,W_{\ell-1})$ has a Dirichlet distribution with parameters $a_1, \ldots, a_\ell$, and write $(W_1,\ldots,W_{\ell-1}) \sim D(a_1,\ldots,a_\ell)$, if its density function is equal to $(\prod_{r=1}^{\ell} \Gamma(a_r))^{-1}\Gamma(\sum_{r=1}^{\ell} a_r) \prod_{r=1}^{\ell-1} x_r^{a_r-1} (1-\sum_{r=1}^{\ell-1} x_r)^{a_\ell-1} I(A)$.

Definition 3.2. Let θ be a positive number. We say that the rva V has a Gamma distribution with parameter θ, and write $V \sim G(\theta)$, if its density function is equal to $(\Gamma(\theta))^{-1}e^{-x}x^{\theta-1}I\{(0,\infty)\}$.

Next, we state without proofs two simple propositions for reference purposes.

Proposition 3.3. Let d be a positive integer and let $V \sim G(d)$. Then V is equal in distribution to the sum of d independent $G(1)$ rva's.

Proposition 3.4. Let $a_1, \ldots, a_\ell$ be positive real numbers, let $V_r \sim G(a_r)$, $r = 1, \ldots, \ell$ be independent rva's, and let $V = \sum_{r=1}^{\ell} V_r$. Then $(V_1V^{-1},\ldots,V_{\ell-1}V^{-1}) \sim D(a_1,\ldots,a_\ell)$.

We are ready now to state and prove the main result of the section. We need some notation. Let $a_{n,r}$, $n = 1, 2, \ldots$, $r = 1, \ldots, \ell$ be ℓ sequences of positive integers converging to ∞ and let $a_n = \sum_{r=1}^{\ell} a_{n,r}$. Further, let $p_r \in (0,1)$, and let $\xi_r \in (-\infty,\infty)$, $r = 1, \dots, \ell$, $\ell \in \{2,3,\dots\}$.

Theorem 3.5. Let $(\theta_{n,1},\dots,\theta_{n,\ell-1}) \sim D(a_{n,1},\dots,a_{n,\ell})$, $n = 1, 2, \dots$. Further, let $\underline{Z} = (Z_1,\dots,Z_{\ell-1})$ be a MVN rve such that $EZ_r = 0$, $EZ_r^2 = p_r(1-p_r)$, $r \in \{1,\dots,\ell-1\}$, and that $EZ_iZ_j = -p_ip_j$, $i \neq j \in \{1,\dots,\ell-1\}$.

Let us assume that

(3.1) $\lim_{n\to\infty} (a_n^{-1}a_{n,r}-p_r)\sqrt{a_n} = \xi_r$, $r = 1, \dots, \ell$, and that

(3.2) $\lim_{n\to\infty} \sum_{r=1}^{\ell} p_r = 1$, $\lim_{n\to\infty} \sum_{r=1}^{\ell} \xi_r = 0$.

Then $((\theta_{n,1}-p_1)\sqrt{a_n},\dots,(\theta_{n,\ell-1}-p_{\ell-1})\sqrt{a_n})$ converges to $\underline{Z}$.

Proof. Let $Y_{q,j} \sim G(1)$, $q = 1, 2, \dots$, $j = 1, \dots, \ell$ be ℓ independent sequences of independent rva's and let $A_{n,r} = a_n^{-1}\sum_{q=1}^{a_{n,r}} Y_{q,r}$, $r = 1, \dots, \ell$, $n = 1, 2, \dots$.

Each of the rva's $A_{n,r}$, $r = 1, \dots, \ell$ consists of i.i.d. summands. Thus, by the normal central limit theorem [Loève (1963), p. 275] the sequences $(A_{n,r}-a_{n,r}a_n^{-1})a_n a_{n,r}^{-1/2}$, $n = 1, 2, \dots$, $r = 1, \dots, \ell$ converge to $N(0,1)$ rva's. We note that the rva's $A_{n,r}$, $r = 1, \dots, \ell$ are independent for $n = 1, 2, \dots$, and that

$$(A_{n,r}-p_r)\sqrt{a_n} = \{(A_{n,r}-a_{n,r}a_n^{-1})(a_n a_{n,r}^{-1/2})\}(a_{n,r}a_n^{-1})^{1/2} + $$
$$+(a_{n,r}a_n^{-1}-p_r)\sqrt{a_n},\ r = 1, \dots, \ell,\ n = 1, 2, \dots .$$

Consequently by Condition (3.1) the rve $((A_{n,1}-p_1)\sqrt{a_n},\dots,(A_{n,\ell}-p_\ell)\sqrt{a_n})$ converges to a normal rve of independent components with means equal to ξ_r and variances equal to p_r, $r = 1, \dots, \ell$ respectively.

Now, let $\beta_1, \dots, \beta_{\ell-1}$ be real numbers such that $\sum_{r=1}^{\ell-1}|\beta_r|>0$ and let $g(x_1,\dots,x_\ell) = (\sum_{r=1}^{\ell-1}\beta_r x_r)(\sum_{r=1}^{\ell} x_r)^{-1}$. By Proposition 3.3 $a_nA_{n,r} \sim G(a_{n,r})$,

$r = 1, \ldots, \ell$ hence by Proposition 3.4 the rva's $\sum_{r=1}^{\ell-1} \beta_r W_{n,r}$ and $g(A_{n,1},\ldots,A_{n,\ell})$ are equal in distribution for $n = 1, 2, \ldots$. We note that the function g has continous and nonvanishing first order partial derivatives at $(p_1,\ldots,p_\ell)$. Thus, by Condition (3.2) $\sum_{r=1}^{\ell} \beta_r W_{n,r}$ converges to a $N(\sum_{r=1}^{\ell-1} \beta_r Q_r, \sum_{r=1}^{\ell-1} \beta_r^2 p_r - (\sum_{r=1}^{\ell-1} \beta_r p_r)^2)$ rva [Rao(1973) p. 387].

Finally, the convergence of $\sum_{r=1}^{\ell-1} \beta_r W_{n,r}$ to a normal rva with the parameters specified above for all real numbers $\beta_1, \ldots, \beta_{\ell-1}$ such that $\sum_{r=1}^{\ell-1} |\beta_r| > 0$ is equivalent to the result of the theorem. [Billingsley (1968), p. 49]. ||

4. The asymptotic behaviour of $X_n(\cdot)$.

Let $I(L(n)) = n\sum_{q=1}^{L(n)}((n-q+1)(X_n(0)+q-1))^{-1}$ and let $\sigma^2(L(n)) = n^2\sum_{q=1}^{L(n)}((n-q+1)(X_n(0)+q-1))^{-2}$, $n = 1, 2, \ldots$. Further, let $X = \sum_{r=1}^{m} X_r$, let $f(z) = a^{-1}(1+X)^{-1}\ln((X+z)(1-z)^{-1}X^{-1})$, $z\in(0,1)$, and let $u(t)$ be the inverse function of $f(z)$, $u(t) = X(-1+\exp at(1+X))(1+X\exp at(1+X))^{-1}$, $t\in(0,\infty)$. Finally, let $\tau^2(t) = (1+X)^{-2}(2t - X(1+X)^{-1} + X^{-1}(1+X)^{-1} + X(1+X)^{-1}\exp at(1+X) - X^{-1}(1+X)^{-1}\exp(-at(1+X)))$, $t\in(0,\infty)$, and let $[y]$ be the largest integer less than equal to y, $y\in(-\infty,\infty)$.

In this section we prove that $\sqrt{n}(X_n(t)-u(t)-X)$ converges to a $N(0,\tau^2(t))$ rva for all $t\in(0,\infty)$ provided Condition (1.10) holds. We later use this result in Section 5.

To prove the main theorem of this section we need the following three lemmas.

Lemma 4.1. Let us assume that condition (1.10) holds and that

$$\lim_{n\to\infty}(n^{-1}L(n)-z)\sqrt{n} = y,\quad z\in(0,1),\ y\in(-\infty,\infty). \tag{4.1}$$

Then

$$\lim_{n\to\infty}(I(L(n))-af(z))\sqrt{n} = y((X+z)(1-z))^{-1} \tag{4.2}$$

Proof. Let $\gamma_n = n^{-1}(nv-[nv])$ and let $\delta_n = n^{-1}([nv]-nv) + n^{-1}X_n(0)-X$, $n = 1, 2, \ldots$. We observe that (a) $I(L(n)) = n\int_0^{L(n)}((n-[v])(X_n(0)+[v]))^{-1}dv = n^2\int_0^{n^{-1}L(n)}((n-[nv])(X_n(0)+[nv]))^{-1}dv = \int_0^{n^{-1}L(n)}((1-v+\gamma_n)(X+v+\delta_n))^{-1}dv$, that (b) $\lim_{n\to\infty} n^{1/2}\gamma_n = 0$, and that (c) $\lim_{n\to\infty} n^{1/2}\delta_n = 0$ [illegible]. it suffices by the mean value theorem and Statements (a), (b) and (c) to show that

$$(4.3)\qquad \lim_{n\to\infty}\left(\int_0^{n^{-1}L(n)}((1-y)(\lambda+y))^{-1}dy\right)\sqrt{n} - v((\lambda+z)(1-z))^{-1};$$

since, $af(z) = \int_0^z ((1-y)(\lambda+y))^{-1}dy$.

Finally, Statement (4.3) follows clearly from the mean value theorem and Condition (4.1). ||

<u>Lemma 4.2.</u> Let us assume that Conditions (1.10) and (4.1) hold. Then

$$(4.4)\qquad \lim_{n\to\infty} n\sigma^2(L(n)) = (1+\lambda)^{-2}\{z\lambda^{-1}(\lambda+z)^{-1}+z(1-z)^{-1}+2af(z)\}.$$

<u>Proof.</u> We observe that $\sigma^2(L(n)) = n^2(\Lambda_n(0)+n)^{-2}\{\sum_{q=1}^{L(n)}(n-q+1)^{-2} + \sum_{q=1}^{L(n)}(\Lambda_n(0)+q-1)^{-2} + 2n^{-1}\ell(L(n))\}$. The conclusion of the lemma follows from Statement (4.2) since, $\lim_n n\sum_{q=1}^{L(n)}(n-q+1)^{-2} = z(1-z)^{-1}$ and since,

$\lim_{n\to\infty} n\sum_{q=1}^{L(n)}(\Lambda_n(0)+q-1)^{-2} = z\lambda^{-1}(\lambda+z)^{-1}$. ||

<u>Lemma 4.3.</u> Let us assume that Conditions (1.10) and (4.1) hold. Then $(S_{n,L(n)}-f(z))\sqrt{n}$ converges to a $N(va^{-1}(\lambda+z)^{-1}(1-z)^{-1},$ $(1+\lambda)^{-2}\{z\lambda^{-1}(\lambda+z)^{-1}+z(1-z)^{-1}+2f(z)\})$ rva.

Proof. Let $V_q \sim G(1)$, $q = 1, 2, \ldots$, be a sequence of independent rva's. Further, let $F_{n,q} = n\{a\sigma(L(n))(n-q+1)(\Lambda_n(0)+q-1)\}^{-1}(V_q-1)$, $q = 1, \ldots, L(n)$, $n = 1, 2, \ldots$, and let $c(L(n)) = a\sigma L(n)n^{-1}\min_{1\le q\le L(n)}\{(n-q+1)(\Lambda_n(0)+q-1)\}$.

By Lemma 2.4(b) the rva's $(S_{n,L(n)}-ES_{n,L(n)})\sigma^{-1}(L(n))$ and $\sum_{q=1}^{L(n)}F_{n,q}$ are equal in distribution for $n = 1, 2, \ldots$. We note that

$$(S_{n,L(n)}-f(z))\sqrt{n} = (S_{n,L(n)}-ES_{n,L(n)})\sigma^{-1}(L(n))(n\sigma^2(L(n)))^{1/2} + a^{-1}(\ell(L(n))-af(z))\sqrt{n}.$$

Thus, to prove the result of the lemma it suffices by Lemmas 4.1 and 4.2 to show that $\sum_{q=1}^{L(n)}F_{n,q}$ converges to a $N(0,a^{-2})$ rva.

Next, we show that $\sum_{q=1}^{L(n)} F_{n,q}$ converges to a $N(0,\alpha^{-2})$ rva. Let $\epsilon>0$ and let $h_n(\epsilon) = \sum_{q=1}^{L(n)} EF_{n,q}^2 I(|F_{n,q}|>\epsilon)$, $n = 1, 2, \ldots$. Clearly, for all $\epsilon>0$ $h_n(\epsilon) \le \alpha^{-2}E((V_1-1)^2 I(|V_1-1|>\epsilon c(L(n))))$. By the dominated convergence theorem we have for all $\epsilon>0$ that $\lim_{n\to\infty} h_n(\epsilon) = 0$ since, $\lim_{n\to\infty} c(L(n)) = \infty$. Thus, $\sum_{q=1}^{L(n)} F_{n,q}$ converges to a $N(0,\alpha^{-2})$ rva by the normal central limit theorem [Loève (1963), p. 280]. ||

We are ready now to prove the main result of this section.

<u>Theorem 4.4</u>. Let us assume that Condition (1.10) holds, and let $t\in(0,\infty)$. Then $\{n^{-1}(X_n(t)-X_n(0))-\mu(t)\}\sqrt{n}$ converges to a $N(0,\tau^2(t))$ rva.

<u>Proof</u>. Let $v\in(-\infty,\infty)$ and let $L(n) = \lfloor(vn^{-1/2}+\mu(t))n\rfloor$, $n = 1, 2, \ldots$. We observe that $P\{(n^{-1}(X_n(t)-X_n(0))-\mu(t))\sqrt{n} > v\} = P\{S_{n,L(n)}<t\} = P\{(S_{n,L(n)}-f(\mu(t))\sqrt{n} <0\}$. Now we note that $L(n)$ satisfies Condition (4.1) with $z = \mu(t)\in(0,1)$. Thus, the conclusion of the theorem follows from Lemma 4.3. ||

Finally, we observe that $(n^{-1}X_n(t)-\mu(t)-\lambda)\sqrt{n} = \{n^{-1}(X_n(t)-X_n(0))-\mu(t)\}\sqrt{n} + (X_n(0)-\lambda)\sqrt{n}$ for $n = 1, 2, \ldots$. Thus, we obtain by Condition (1.10) and Theorem 4.4 the following.

<u>Corollary 4.5</u>. Let us assume that Condition (1.10) holds, and let $t\in(0,\infty)$. Then $\{n^{-1}X_n(t)-\mu(t)-\lambda\}\sqrt{n}$ converges to a $N(0,\tau^2(t))$ rva.

5. Main Results.

Let $\gamma_r = \lambda_r \lambda^{-1}$, $r = 1, \ldots, m$, and let z be in the interval $(0,1)$. In this section we prove under Conditions (1.10) and (1.11) our main results. First, we show that the rve $\{(L^{-1}(n)\sum_{q=1}^{L(n)} I(\xi_{n,q}=r)-\gamma_r)\sqrt{L(n)}, r=1,\ldots,m\}$ converges to a MVN rve. Next, we show that the rve $\{(n^{-1}X_{n,r}(t)-\gamma_r\mu(t)-\lambda_r)\sqrt{n}, r=1,\ldots,m\}$ converges to a MVN rve for all $t\in(0,\infty)$.

For the sake of completeness we present the following definition.

Definition 5.1. Let Y_q, $q = 1, 2, \ldots$, be a sequence of rva's. We say that Y_q, $q = 1, 2, \ldots$, is an exchangeable sequence of rva's if for all positive integers n and all permutations Π of the set $\{1,\ldots,n\}$ the rve's $\{Y_1, \ldots, Y_n\}$ and $\{Y_{\Pi(1)},\ldots,Y_{\Pi(n)}\}$ are equal in distribution.

Let n be a positive integer. From Equation (2.2) and the extension of Equation (2.3) to all $k\in\{2,3,\ldots\}$ we conclude that $\xi_{n,q}$, $q = 1, 2, \ldots$, is an exchangeable sequence of rva's. Thus, by DeFinetti's theorem [Feller (1966), p. 225] there is a rve $\underline{W}_n=(W_{n,1},\ldots,W_{n,m-1})$ [illegible] $(X_{n,1}(0),\ldots,X_{n,m}(0))$ such that

(5.1) The conditional sequence of rva's $\xi_{n,q}|\underline{W}_n$, $q = 1, 2, \ldots$, consists of i.i.d. rva's, and that

(5.2) $P\{\xi_{n,1}=r|\underline{W}_n\} = W_{n,r}$, $r = 1, \ldots, m-1$.

To prove the first main result we need some notation and two lemmas. Let ϕ be the distribution function of a $N(0,1)$ rva, let $U_{n,r}=(W_{n,r}-\gamma_r)\sqrt{n}$, $r = 1, \ldots, m-1$, $n = 1, 2, \ldots$, and let $\beta_1, \ldots, \beta_{m-1}$ be real numbers. Further, let $I_{n,1}(s) = I(|U_{n,r}|\le s, r=1,\ldots,m-1)$, $s\in(0,\infty)$, let $\sigma_n^2(\beta_1,\ldots,\beta_{m-1}) = \sum_{r=1}^{m-1} \beta_r^2 W_{n,r}-(\sum_{r=1}^{m-1} \beta_r W_{n,r})^2$, and let $\sigma^2(\beta_1,\ldots,\beta_{m-1})=\sum_{r=1}^{m-1} \beta_r^2\gamma_r-(\sum_{r=1}^{m-1} \beta_r\gamma_r)^2$.

Throughout we assume that $\beta = \sum_{r=1}^{m-1}|\beta_r| > 0$.

Lemma 5.2. Let $R(n)\in\{1,\ldots,n\}$, $n = 1, 2, \ldots$, be a sequence of rva's independent of $\underline{W}_n$ for $n = 1, 2, \ldots$, let $I_{n,2}(w) = I(|n^{-1}R(n)-z|\sqrt{n} \le w)$, and let $B(R(n),x)=(x-R(n)\sum_{r=1}^{m-1}\beta_r W_{n,r})R^{-1/2}(n)\sigma_n^{-1}(\beta_1,\ldots,\beta_{m-1})$, $n = 1, 2, \ldots$, $x\in(-\infty,\infty)$, $w\in(0,\infty)$. Then for all real numbers s and w

$$(5.3) \qquad \lim_{n\to\infty} \sup_{-\infty<x<\infty} |EI_{n,2}(w)I_{n,1}(s)(I(\sum_{r=1}^{m-1}\beta_r\sum_{q=1}^{R(n)}I(\xi_{n,q}=r)\le x) - \Phi(B(R(n),x))| = 0.$$

Proof. Let $H_{n,q}= \sum_{r=1}^{m-1} \beta_r I(\xi_{n,q}=r)$, $q = 1, 2, \ldots$. By Statement (5.1) and by the independence of $R(n)$ and $\underline{W}_n$ the conditional rva's $H_{n,q}|\underline{W}_n,R(n)$, $q = 1, 2, \ldots$, are i.i.d. By Statement (5.2) and by the independence of $R(n)$ and $\underline{W}_n$ $E(H_{n,1}|\underline{W}_n,R(n)) = \sum_{r=1}^{m-1}\beta_r W_{n,r}$ and $Var(H_{n,1}|\underline{W}_n,R(n)) = \sigma_n^2 (\beta_1,\ldots\beta_{m-1})$, $n = 1, 2, \ldots$. We note that $\sum_{q=1}^{R(n)}H_{n,q}=\sum_{r=1}^{m-1}\beta_r\sum_{q=1}^{R(n)}I(\xi_{n,q}=r)$ and that $|H_{n,1}-E(H_{n,1}|\underline{W}_n,R(n))|\le 2\beta$, $n = 1, 2, \ldots$. Thus, by the Berry-Esseen bound [Loève (1963), p. 288]

$$(5.4) \qquad \sup_{-\infty<x<\infty} |P\{\sum_{r=1}^{m-1}\beta_r\sum_{q=1}^{R(n)}I(\xi_{n,q}=r)\le x|\underline{W}_n,R(n)\}-\Phi(B(R(n),x))| \le 2c\beta R^{-1/2}(n)\sigma_n^{-3}(\beta_1,\ldots,\beta_{m-1}),$$ where c is a positive constant.

Now, $E\, I_{n,2}(w)I_{n,1}(s)I(\sum_{r=1}^{m-1}\beta_r\sum_{q=1}^{R(n)}I(\xi_{n,q}=r)\le x) = E(I_{n,2}(w)I_{n,1}(s)E\{I(\sum_{r=1}^{m-1}\beta_r\sum_{q=1}^{R(n)}I(\xi_{n,q}=r)\le x)|\underline{W}_n,R(n)\})$ for $n = 1, 2, \ldots$, s, $w\in(0,\infty)$. Thus, to prove Statement (5.3) it suffices by Inequality (5.4) to show that for all s, $w\in(0,\infty)$

$$(5.5) \qquad \lim_{n\to\infty} EI_{n,2}(w)R^{-1/2}(n)I_{n,1}(s)\sigma_n^{-3}(\beta_1,\ldots\beta_{m-1}) = 0.$$

Finally, we prove Statement (5.5). Let s, $w\in(0,\infty)$. We note that $\sigma_n^2 (|\beta_1|,\ldots,|\beta_{m-1}|)\ge\sum_{r=1}^{m-1} \beta_r^2(\gamma_r-sn^{-1/2})-(\sum_{r=1}^{m-1}|\beta_r|(\gamma_r+sn^{-1/2}))^2$ on the set

$\{|U_{n,r}|\le s, r=1,\ldots,m-1\}$, and that $\lim_{n\to\infty}\{\sum_{r=1}^{m-1}\beta_r^2(\gamma_r - sn^{-1/2}) - (\sum_{r=1}^{m-1}|\beta_r|(\gamma_r+sn^{-1/2})^2\} =$ $= \sigma^2(|\beta_1|,\ldots,|\beta_{m-1}|)\le\sigma^2(\beta_1,\ldots,\beta_{m-1})$. Now for n sufficiently large $0 \le I_{n,2}(w)R^{-1/2}(n) \le n^{-1/2}(z-n^{-1/2}w)^{-1/2}$ and $I_{n,1}(s)\,\sigma_n^{3}(\beta_1,\ldots,\beta_{m-1})$ is bounded from above. Consequently Statement (5.5) follows. ||

<u>Lemma 5.3</u>. Let $\underline{Z} = \{Z_1,\ldots,Z_{m-1}\}$ be a MVN rve such that $EZ_r = 0$, $EZ_r^2 = \gamma_r(1-\gamma_r)$, $r\in\{1,\ldots,m-1\}$ and that $EZ_iZ_j = -\gamma_i\gamma_j$, $i\ne j\in\{1,\ldots,m-1\}$. Let us assume that Condition (1.10) holds. Then $\{U_{n,1},\ldots,U_{n,m-1}\}$ converges to $\underline{Z}$.

<u>Proof</u>. We note that $(X_n^{-1}(0)X_{n,r}(0)-\gamma_r)\sqrt{X_n(0)} =$ $= \sqrt{nX_n^{-1}(0)}\{(n^{-1}X_{n,r}(0)-\lambda_r)\sqrt{n}-\gamma_r(n^{-1}X_n(0)-\lambda)\sqrt{n}\}$, $n = 1, 2, \ldots$. Thus, by Condition (1.10) $\lim_{n\to\infty}(X_n^{-1}(0)X_{n,r}(0)-\gamma_r)\sqrt{X_n(0)} = 0$ for $r = 1, \ldots, m$. The result of the lemma follows now from Theorem 3.5 since, $\sum_{r=1}^{m}\gamma_r = 1$. ||

We are ready now to prove our first main result.

<u>Theorem 5.4</u>. Let us assume that Condition (1.11) holds. Let $\{Z_1,\ldots,Z_m\}$ be a MVN rve such that $EZ_r = 0$, $EZ_r^2 = (1+z)\gamma_r(1-\gamma_r)$, $r\in\{1,\ldots,m\}$ and that $EZ_iZ_j = -(1+z)\gamma_i\gamma_j$, $i\ne j\in\{1,\ldots,m\}$. Then the rve $\{(L^{-1}(n)\sum_{q=1}^{L(n)}I(\xi_{n,q}=r)-\gamma_r)\sqrt{L(n)}, r=1,\ldots,m\}$ converges to $\underline{Z}$.

<u>Proof</u>. Let $M_{n,r} = \{L^{-1}(n)\sum_{q=1}^{L(n)}I(\xi_{n,q}=r)-\gamma_r\}\sqrt{L(n)}$, $r = 1, \ldots, m$, $n = 1, 2, \ldots$, let $M_n = \sum_{r=1}^{m-1}\beta_r M_{n,r}$, $n = 1, 2, \ldots$, let $M\sim N(0,(1+z)\sigma^2(\beta_1,\ldots,\beta_{m-1}))$ rva, and let $e_n(x) = (L^{-1/2}(n)x+\sum_{r=1}^{m-1}\beta_r\gamma_r)L(n)$, $n = 1, 2, \ldots$, $x\in(-\infty,\infty)$.

Since, $\sum_{r=1}^{m} M_{n,r} = 0$ to prove the result of the lemma it suffices to show that $\{M_{n,1},\ldots,M_{n,m-1}\}$ converges to $\{Z_1,\ldots Z_{m-1}\}$. Further, to prove the preceeding statement it is enough to show that M_n converges to M for all real numbers $\beta_1,\ldots,\beta_{m-1}$ such that $\sum_{r=1}^{m-1}|\beta_r|>0$ [Billingsley (1968), p. 49]. Next, we show that M_n converges to M.

We observe that $|EI(M_n \le x) - EI_{n,1}(s)I(M_n \le x)| \le 2P\{|U_{n,r}| > s, r=1,\dots,m-1\}$ for $n = 1, 2, \dots,$ and for $s \in (0,\infty)$. By Lemma 5.3 $\lim_{s\to\infty} \lim_{n\to\infty} P\{|U_{n,r}| > s, r=1,\dots,m-1\} = 0$. Thus, to show that M_n converges to M it suffices to prove that for all $x \in (-\infty,\infty)$

$$\lim_{s\to\infty} \lim_{n\to\infty} EI_{n,1}(s)I(M_n \le x) = P\{M \le x\}. \tag{5.6}$$

Next, we apply Lemma 5.2 for a sequence $R(n)$, $n = 1, 2, \dots,$ of rva's degenerate at $L(n)$. Let $w \in (0,\infty)$ then $I_{n,2}(w) = 1$ for n sufficiently large. Hence, we conclude from Statement (5.3) that $\lim_{s\to\infty} \lim_{n\to\infty} |EI_{n,1}(s)I(M_n \le x) - EI_{n,1}(s)\phi\{B(L(n),e_n(x))\}| = 0$. Thus, to prove Statement (5.6) it is enough to show that for all $x \in (-\infty,\infty)$

$$\lim_{s\to\infty} \lim_{n\to\infty} EI_{n,1}(s)\phi\{B(L(n),e_n(x))\} = P\{M \le x\}. \tag{5.7}$$

Now, we observe that $|E(I_{n,1}(s)-1)\phi\{B(L(n),e_n(x)\}| \le 2P\{|U_{n,r}| > s, r=1,\dots,m-1\}$ for $n = 1, 2, \dots,$ and for $s \in (0,\infty)$. Thus, by Lemma 5.3 to prove Statement (5.7) it suffices to show that for all $x \in (-\infty,\infty)$

$$\lim_{n\to\infty} E\phi\{B(L(n),e_n(x))\} = P\{M \le x\}. \tag{5.8}$$

Finally, we prove Statement (5.8). Let Z be a $N(0,1)$ rva independent of $\underline{W}_n$ for $n = 1, 2, \dots$. We note that $E\phi\{B(L(n),e_n(x))\} = P\{Z \le B(L(n),e_n(x))\} = P\{Z\sigma_n(\beta_1,\dots,\beta_{m-1}) + (\sum_{r=1}^{m-1} \beta_r U_{n,r}) L^{1/2}(n) n^{-1/2} \le x\}$. By Lemma 5.3 $\sum_{r=1}^{m-1} \beta_r U_{n,r}$ converges to a $N(0,\sigma^2(\beta_1,\dots,\beta_{m-1}))$ rva. By Lemma 5.3 and by the dominated convergence theorem $Z\sigma_n(\beta_1,\dots,\beta_{m-1})$ converges in probability to $Z\sigma(\beta_1,\dots,\beta_{m-1})$. Since, Z is independent of $\underline{W}_n$ for $n = 1, 2, \dots,$ Statement (5.8) follows. ||

Next, we prove our second main result. We need the following lemma.

<u>Lemma 5.5</u>. Let t be a positive real number and let n be a positive integer.

Then the conditional rve $\{X_{n,r}(t)-X_{n,r}(0), r=1,\ldots,m\}|X_n(t)$ and the rve $\{\sum_{q=1}^{X_n(t)-X_n(0)} I(\xi_{n,q}=r), r=1,\ldots,m\}$ are equal in distribution.

Proof. Let $k_1, \ldots, k_m$ be nonnegative integers such that $k = \sum_{r=1}^m k_r \in \{0,\ldots,n\}$. By Equation (2.1)

$$P\{X_{n,r}(t)-X_{n,r}(0)=k_r, r=1,\ldots,m, X_n(t)-X_n(0)=k\} =$$
$$=P\{S_{n,k}\le t<S_{n,k+1}, \sum_{q=1}^k I(\xi_{n,q}=r)=k_r, r=1,\ldots,m\}.$$

By Lemma 2.4(a)

$$P\{S_{n,k}\le t<S_{n,k+1}, \sum_{q=1}^k I(\xi_{n,q}=r)=k_r, r=1,\ldots,m\} =$$
$$= P\{S_{n,k}\le t<S_{n,k+1}\}P\{\sum_{q=1}^k I(\xi_{n,q}=r)=k_r, r=1,\ldots,m\} =$$
$$= P\{X_n(t)-X_n(0)=k\}P\{\sum_{q=1}^k I(\xi_{n,q}=r)=k_r, r=1,\ldots,m\}.$$

Consequently the result of the lemma follows. ||

Theorem 5.6. Let us assume that Condition (1.10) holds, and let $t\in(0,\infty)$. Further, let $\underline{Z}(t) = \{Z_1(t),\ldots,Z_m(t)\}$ be a MVN rve such that $EZ_r(t) = 0$, $EZ_r^2(t) = \tau^2(t)\gamma_r^2 + \mu(t)(1+\mu(t))\gamma_r(1-\gamma_r)$, $r\in\{1,\ldots,m\}$, and that $EZ_i(t)Z_j(t) = \{\tau^2(t)-\mu(t)(1+\mu(t)\}\gamma_i\gamma_j$, $i\ne j\in\{1,\ldots,m\}$. Then the rve $\{(n^{-1}X_{n,r}(t)-\gamma_r\mu(t)-\lambda_r)\sqrt{n}, r=1,\ldots,m\}$ converges to $\underline{Z}(t)$.

Proof. Let $\nu_1, \ldots, \nu_m$ be real numbers such that $\sum_{r=1}^m |\nu_r|>0$, and let $e_{n,1}(x) = (xn^{-1/2}+\mu(t)\sum_{r=1}^m \nu_r\gamma_r)n$, $n = 1, 2, \ldots$, $x\in(-\infty,\infty)$. Further, let $\Delta_n = \sqrt{n}\sum_{r=1}^m \nu_r\{(X_{n,r}(t)-X_{n,r}(0))n^{-1}-\gamma_r\mu(t)\}$, $n = 1, 2, \ldots$, let $\Delta\sim N(0,\sum_{r=1}^m \nu_r^2(\gamma_r^2\tau(t)+\mu(t)(1+\mu(t))\gamma_r(1-\gamma_r)+2\sum_{1\le i<j\le m}\nu_i\nu_j(\tau^2(t)-\mu(t)(1-\mu(t)))$, let $V_n(t) = \{(X_n(t)-X_n(0))n^{-1}-\mu(t)\}\sqrt{n}$, and let $R(n) = (n^{-1/2}V_n(t)+\mu(t))n$. Finally, let $B(R(n),x)$ be as in Lemma 5.2, and let $I_{n,3}(w) = I(|V_n(t)|\le w)$, $n = 1, 2, \ldots$, $w\in(0,\infty)$.

By Lemma 2.4(a) the rve $\{\xi_{n,1},\ldots,\xi_{n,n}\}$ and the rva's $X_n(t)$, $t\in(0,\infty)$ are independent for $n = 1, 2, \ldots$. Thus, $X_n(t)$, $t\in(0,\infty)$ and $\underline{W}_n$ are independent for $n = 1, 2, \ldots$.

We note that

$(n^{-1}\lambda_{n,r}(t)-\gamma_r\mu(t)-\lambda_r)\sqrt{n} = (X_{n,r}(0)-\lambda_r)\sqrt{n} + \{n^{-1}(X_{n,r}(t)-X_{n,r}(0))-\gamma_r\mu(t)\}\sqrt{n}$,

$r = 1, \ldots, m$. Thus, to prove the result of the theorem it suffices by Condition (1.10) to show that the rve $\{(n^{-1}(X_{n,r}(t)-X_{n,r}(0))-\gamma_r\mu(t))\sqrt{n}, r=1,\ldots,m\}$ converges to $\underline{Z}(t)$. To prove the preceeding statement it is enough to show that Δ_n converges to Δ for all real numbers $\nu_1,\ldots,\nu_m$ such that $\sum_{r=1}^{m}|\nu_r|>0$ [Billingsley (1968), p. 49]. Since, by Theorem 4.4 Δ_n converges to Δ for $\nu_1 = \nu_2 \ldots = \nu_m \neq 0$ it suffices to prove that Δ_n converges to Δ for $\nu_1, \ldots, \nu_m$ such that $\sum_{r=1}^{m}|\nu_m|>0$ and that $\sum_{r=1}^{m-1}|\nu_r-\nu_m|>0$. Let us denote $\nu_r-\nu_m$ by β_r, $r = 1, \ldots, m-1$ respectively. Next, we prove that Δ_n converges to Δ. We assume throughout that $\sum_{r=1}^{m-1}|\beta_r|>0$.

We note that $|EI(\Delta_n\le x)(1-I_{n,1}(s)I_{n,3}(w)|\le P\{|V_n(t)|>w\}+P\{|U_{n,r}|>s, r=1,\ldots,m-1\}$. Thus, to prove that Δ_n converges to Δ it suffices by Theorem 4.4 and Lemma 5.3 to show that for all $x\in(\infty,\infty)$

(5.9) $$\lim_{\substack{s\to\infty\\ w\to\infty}} EI(\Delta_n\le x)I_{n,1}(s)I_{n,3}(w)=P\{\Delta\le x\}.$$

Next, we note that by Lemma 5.5 $EI(\Delta_n\le x)I_{n,1}(s)I_{n,3}(w) =$

$= E\{I_{n,1}(s)I_{n,3}(w)E(I(\Delta_n\le x)|\underline{W}_n, R(n))\}=EI_{n,1}(s)I_{n,3}(w)I(\sum_{r=1}^{m}\nu_r\sum_{q=1}^{R(n)}I(\xi_{n,q}=r)\le e_{n,1}(x))$.

Since, $\sum_{r=1}^{m}I(\xi_{n,q}=r) = 1$, $q = 1, 2, \ldots, n = 1, 2, \ldots$, we obtain that

$EI_{n,1}(s)I_{n,3}(w)\ I(\Delta_n\le x)=EI_{n,1}(s)I_{n,3}(w)I(\sum_{r=1}^{m-1}\beta_r\sum_{q=1}^{R(n)}I(\xi_{n,q}=r)\le e_n(x)-\nu_m R(n))$.

Thus, to prove Statement (5.9) it is enough to show by Lemma 5.2 that

(5.10) $$\lim_{\substack{s\to\infty\\ w\to\infty}}\ \lim_{n\to\infty} EI_{n,1}(s)I_{n,3}(w)\phi\{B(R(n), e_{n,1}(x)-\nu_m R(n)\}=P\{\Delta\le x\}.$$

Now, $|E(I_{n,1}(s)I_{n,3}(w)-1)\phi\{B(R(n), e_{n,1}(x)-\nu_m R(n)\}\le P\{|V_n(t)|>w\} +$ $+P\{|U_{n,r}|>s, r=1,\ldots,m-1\}$. Hence, to prove Statement (5.10) it suffices by

Theorem 4.4 and Lemma 5.3 to show that

$$(5.11)\qquad \lim_{n\to\infty} E\phi\{B(R(n),e_{n,1}(x)-\nu_m R(n))\}=P\{\Delta\le x\}.$$

Finally, we prove Statement (5.11). Let Z be a $N(0,1)$ rva independent of $V_n(t)$ and $\underline{W}_n$ for $n = 1, 2, \dots$. Now $E\phi\{B(R(n),e_{n,1}(x)-\nu_m R(n))\} =$ $= P\{Z\le B(R(n),e_{n,1}(x)-\nu_m R(n)\}=P\{Z\sigma_n(\beta_1,\dots,\beta_{m-1})R^{1/2}(n)n^{-1/2}+n^{-1}R(n)\sum_{r=1}^{m-1}\beta_r U_{n,r}+$ $+(\sum_{r=1}^{m}\nu_r\gamma_r)V_n(t)\le x\}$. By Lemma 5.3 and Theorem 4.4 $n^{-1}R(n)\sum_{r=1}^{m-1}\beta_r U_{n,r}$ converges to a $N(0,\mu^2(t)\sigma^2(\beta_1,\dots,\beta_{m-1}))$ rva. By Lemma 5.3, by the dominated convergence theorm, and by Theorem 4.4 $Z\sigma_n(\beta_1,\dots,\beta_{m-1})R^{1/2}(n)n^{-1/2}$ converges in probability to $Z\sigma(\beta_1,\dots,\beta_{m-1})\mu^{1/2}(t)$. By Theorem 4.4 $V_n(t)$ converges to a $N(0,\tau^2(t))$ rva. Since, $V_n(t)$ and Z are independent of $\underline{W}_n$ for $n = 1, 2, \dots$, Statement (5.11) follows. ||

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20. ABSTRACT

We consider a group of n susceptible individuals who are exposed to m contagious diseases. The progress of the epidemic among the individuals is modeled by a stochastic process $\underset{\sim}{X}_n(t) \equiv (X_{n,1}(t), \ldots, X_{n,m}(t))$, t in $(0, \infty)$. The components of $\underset{\sim}{X}_n(t)$ describe the number of infective individuals with the respective disease at time t.

For a class of epidemic models named symmetric m-dimensional simple epidemics [Billard, Lacayo and Langberg (1979)] we establish, with a suitable standartization, the asymptotic normal convergence of $\underset{\sim}{X}_n(t)$ as $n \to \infty$ for t in $(0, \infty)$.